ON ZEROS OF THE ALEXANDER POLYNOMIAL OF AN ALTERNATING KNOT

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Abstract. We prove that for any zero α of the Alexander polynomial of a two-bridge knot, $-3 < \mathrm{Re}(\alpha) < 6$. Furthermore, for a large class of two-bridge knots we prove $-1 < \mathrm{Re}(\alpha)$.

Contents

1. Introduction

In 2002 Jim Hoste made the following conjecture based on his extensive computer experiment:

Conjecture 1. (J. Hoste, 2002) Let K be an alternating knot and $\Delta_K(t)$ be its Alexander polynomial. Let α be a zero of $\Delta_K(t)$. Then $\mathrm{Re}(\alpha) > -1$.

This conjecture is known to be true for some classes of alternating knots.

1) If K is a special alternating knot, then all zeros of its Alexander polynomial lie on a unit circle ([M2],[L],[T]), and $\Delta_K(-1) \neq 0$, so Conjecture 1 holds.

2) If α is a real zero of the Alexander polynomial $\Delta_K(t)$ of an alternating knot K, then $\alpha > 0$, since the coefficients of the Alexander polynomial of an alternating knot have alternating signs ([C],[M1]). Therefore, if all zeros are real, then K satisfies Conjecture 1.

3) Any knot K with $\deg \Delta_K(t) = 2$ satisfies $-1 < \mathrm{Re}(\alpha) < 3$. Any alternating knot K with $\deg \Delta_K(t) = 4$ satisfies Conjecture 1.

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The problem of finding a lower or upper bound of the real part of zeros of the Alexander polynomial is reduced to a problem of showing the stability of the matrix associated to a Seifert matrix U of a knot. Then we apply a well known Lyapunov theorem on the stability of matrices. This approach, described in detail in section 2 below, is particularly successful for two-bridge knots. A two-bridge knot $K = K(r)$ is identified by a rational number r. We use an even negative continued fraction expansion $r = [2a_1, 2a_2, \dots, 2a_m]$ to construct a knot diagram $\Gamma(K(r))$, a Seifert surface F and its Seifert matrix U.

Throughout the paper by a two-bridge knot we will mean a two-bridge knot or a two-component two-bridge link, and its Alexander polynomial is defined by $\Delta_{K(r)} = \det(Ut - U^T)$ (see [BZ]).

In this paper we prove the following theorems:

Theorem 1. Let $K(r)$ be a two-bridge knot, $\Delta_K(t)$ be its Alexander polynomial and α be a zero of $\Delta_K(t)$. Then

$$-3 < \mathrm{Re}(\alpha) < 6.$$

Theorem 2. Let $K(r)$ be a two-bridge knot, $r = [2a_1, 2a_2, \dots, 2a_m]$. If $a_i a_{i+1} < 0$ for $i = 1, 2, \dots, m-1$, then all zeros are real, hence the conjecture holds.

Theorem 3. Let $K(r)$ be a two-bridge knot, $r = [2a_1, 2a_2, \dots, 2a_m]$. If among $a_1, \dots, a_m$ there are no two consecutive 1 or -1 (namely, $a_i a_{i+1} \neq 1$ for $i = 1, \dots, m-1$), then the conjecture holds. If moreover $|a_i| > 1$ for $i = 1, 2, \dots, m$, then $-1 < \mathrm{Re}\,\alpha < 3$.

It is known that $K(r)$ is fibered if and only if $|a_j| = 1$ for all j.

Theorem 4. Let $K(r)$ be a fibered two-bridge knot with

$$r = [\underbrace{2, \dots, 2,}_{k_1} \ \underbrace{-2, \dots, -2,}_{k_2} \ \dots, \ \underbrace{(-1)^{m-1}2, \dots, (-1)^{m-1}2}_{k_m}] \, .$$

If $k_j = 1$ or 2 for all j, then the conjecture holds.

Theorem 5. Let $K(r)$ be a two-bridge knot, $r = r(m, c) = [2c, -2c, \dots, (-1)^{m-1}2c]$, $c > 0$, $m \geq 1$. Then all zeros of $\Delta_{K(r)}$ satisfy inequality:

$$\left(\frac{\sqrt{1 + c^2} - 1}{c}\right)^2 < \alpha < \left(\frac{\sqrt{1 + c^2} + 1}{c}\right)^2.$$

For non-alternating knots there are no such bounds.

Example 1. Let $\Delta_K(t) = 1 + at - (2a+1)t^2 + at^3 + t^4$, $a > 0$. Since $\Delta_K(-(a+1)) < 0$, there is a zero α of $\Delta_K(t)$ such that $\mathrm{Re}(\alpha) < -a - 1$. K is not alternating.

Example 2. Let $\Delta_K(t) = 1 - 2at + (4a-1)t^2 - 2at^3 + t^4$, $a \geq 4$. Then $\Delta_K(a) < 0$ and hence, there exists a zero α such that $\alpha > a$. K is not alternating. In fact, if K is alternating, then K is fibered and since $\deg \Delta_K(t) = 4$, K has at most 8 crossings. However, such an alternating knot (including non-prime alternating knots) does not exist in the table if $a \geq 4$ (see [BZ]).

2. Stability of matrices and Lyapunov theorem

Let K be an alternating knot (or link) and $\Delta_K(t) = c_0 + c_1 t + c_2 t + \ldots + c_n t^n$, $c_n \neq 0$ be its Alexander polynomial. Let A be a companion matrix of $\Delta_K(t)$ i.e. $\Delta_K(t) = c_n \det(tE - A)$. The eigenvalues of A are the zeros of $\Delta_K(t)$. We have

$$\mathrm{Re}(\alpha) > -1 \Longleftrightarrow \mathrm{Re}(-(1+\alpha)) < 0.$$

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be all zeros of $\Delta_K(t)$ (= all eigenvalues of A). Then it is easy to see that $-(1+\alpha_1), -(1+\alpha_2), \ldots, -(1+\alpha_n)$, are eigenvalues of $-(E+A)$. To prove that all eigenvalues of a matrix have negative real parts, we apply the Lyapunov theorem:
Let M be a real $n \times n$ matrix. Consider a linear vector differential equation

$$\dot{\mathbf{x}} = M\mathbf{x}.$$

It is a known theorem in ODE that all solutions $\mathbf{x}(t) \in \mathbb{R}^n$ of it are stable, namely $\mathbf{x}(t) \longrightarrow 0$ as $t \longrightarrow \infty$, if and only if all eigenvalues of M have negative real parts. In this case M is called stable.

Theorem (Lyapunov). [G] All eigenvalues of M have negative real parts if and only if there exists a symmetric positive definite matrix V such that

$$VM + M^T V = -W, \quad \text{where } W \text{ is positive definite.}$$

Hence K satisfies Conjecture 1 if there exists a positive definite matrix V such that

$$V(E+A) + (E+A^T)V = W \text{ is positive definite.} \tag{2.1}$$

Similarly to (2.1), all zeros of $\Delta_K(t)$ satisfy $-k < \mathrm{Re}(\alpha)$ if and only if $-(kE+A)$ is stable, i.e there exists a positive definite matrix V such that

$$V(kE+A) + (kE+A^T)V = W \text{ is positive definite.}$$

Further, all zeros of $\Delta_K(t)$ satisfy $\mathrm{Re}(\alpha) < q$ if and only if $A - qE$ is stable, i.e. there exists a positive definite matrix V such that

$$V(qE-A) + (qE-A^T)V = W \text{ is positive definite.}$$

To prove that a matrix is positive definite we use the following lemma.

Lemma 1. (Positivity Lemma)

$$\text{Let } N = \begin{bmatrix} a_{11} & a_{12} & & & \\ a_{21} & a_{22} & a_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1,n} \\ & & & a_{n,n-1} & a_{n,n} \end{bmatrix} \text{ be a real symmetric matrix.}$$

Suppose that for $1 \leq j \leq n$
(i) $a_{j,j} > 0$, $a_{j,j-1}, a_{j,j+1} \neq 0$, and all non-specified entries are 0.
(ii) $a_{j,j} \geq |a_{j,j-1}| + |a_{j,j+1}|$,
(iii) there exists i such that $a_{i,i} > |a_{i,i-1}| + |a_{i,i+1}|$.
Then N is positive definite.

The proof is by induction.

3. Two-bridge knots

Let $K(r), 0 < r = \beta/\alpha < 1, \quad 0 < \beta < \alpha$, be a two-bridge knot or a (two-component) two-bridge link of type (α, β). We can assume one of α and β is even. Consider an even (negative) continued fraction expansion of r :

$$r = \beta/\alpha = \cfrac{1}{2a_1 - \cfrac{1}{2a_2 - \ddots - \cfrac{1}{2a_m}}} = [2a_1, 2a_2, \ldots, 2a_m].$$

This expansion is unique. We obtain from it a knot or a link diagram $\Gamma(K(r))$ of $K(r)$. (see Fig.1)

$K(r)$ is a knot $\qquad$ $K(r)$ is a link

$m = 0 (\mathrm{mod}\, 2)$ $\qquad$ $m = 1 (\mathrm{mod}\, 2)$

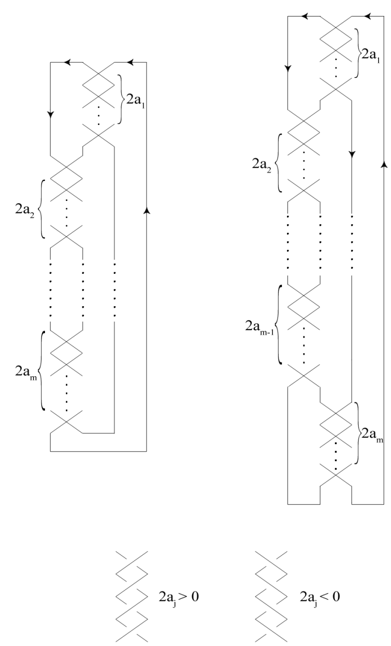

Figure 1.

The following facts are well known:

(1) $K(r)$ is special alternating if and only if $a_1, a_2, \ldots, a_m$ are either all positive or all negative.
(2)$K(r)$ is fibered if and only if $|a_j| = 1$ for all j.
(3)$\Gamma(K(r))$ is an alternating diagram if and only if $a_j a_{j+1} < 0$ for $j = 1, 2, \ldots m-1$.
(4) $\Gamma(K(r))$ gives a minimal genus Seifert surface F for $K(r)$ (see Fig.2).

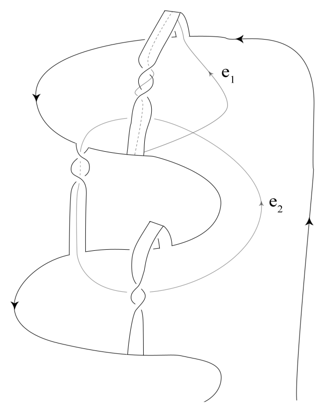

Figure 2. Seifert surface F.

We use this Seifert surface to calculate a Seifert matrix $U = (u_{ij})$ of K, $u_{ij} = lk(e_i^{\#}, e_j)$, $i, j = 1, \ldots, m$. For the fragment of F with only two bands with (half)twists $2a_1$ and $2a_2$ we have

$$\begin{array}{ll} lk(e_1^{\#}, e_1) = a_1, & lk(e_1^{\#}, e_2) = 0, \\ lk(e_2^{\#}, e_1) = -1, & lk(e_2^{\#}, e_2) = a_2, \end{array}$$

and in general, it is not difficult to see that for a two-bridge knot $K = [2a_1, 2a_2, \ldots, 2a_m]$ a Seifert matrix corresponding to the surface F is:

(3.1)

$$U = \begin{bmatrix} a_1 & 0 & & & & \\ -1 & a_2 & 1 & & & \\ & 0 & a_3 & 0 & & \\ & & -1 & a_4 & 1 & \\ & & & & \ddots & \\ & & & & -1 & a_m \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_1 & 0 & & & & \\ -1 & a_2 & 1 & & & \\ & 0 & a_3 & 0 & & \\ & & -1 & a_4 & 1 & \\ & & & & \ddots & \\ & & & & 0 & a_m \end{bmatrix}$$

(depending on m being even or odd, respectively), where all non-specified entries are 0. The Alexander polynomial of K is $\Delta_K(t) = \det(tU - U^T)$. So $A = U^{-1}U^T$

is a companion matrix for $\Delta_K(t)$. We have

$$U^{-1} = \begin{bmatrix} \frac{1}{a_1} & & & & \dots 0 & \\ \frac{1}{a_1a_2} & \frac{1}{a_2} & -\frac{1}{a_2a_3} & & & \\ & & \frac{1}{a_3} & & & \\ & & \frac{1}{a_3a_4} & \frac{1}{a_4} & -\frac{1}{a_4a_5} & \\ & 0 & & & \ddots & \end{bmatrix} \tag{3.2}$$

and $U^{-1}U^T =$

(3.3)

$$= \begin{bmatrix} 1 & -\frac{1}{a_1} & 0 & & \dots & & & \\ \frac{1}{a_2} & 1-\frac{1}{a_1a_2}-\frac{1}{a_2a_3} & -\frac{1}{a_2} & \frac{1}{a_2a_3} & 0 & \dots & & \\ 0 & \frac{1}{a_3} & 1 & -\frac{1}{a_3} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{a_3a_4} & \frac{1}{a_4} & 1-\frac{1}{a_3a_4}-\frac{1}{a_4a_5} & -\frac{1}{a_4} & \frac{1}{a_4a_5} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{a_5} & 1 & -\frac{1}{a_5} & 0 & \dots \\ & & & \dots & & & & \end{bmatrix}$$

The last row of A is $[0,\dots,0,\frac{1}{a_{m-1}a_m},\frac{1}{a_m},1-\frac{1}{a_{m-1}a_m}]$ if m is even, and $[0,\dots,0,\frac{1}{a_m},1]$ if m is odd.

$A = U^{-1}U^T$ is a companion matrix for the Alexander polynomial of the two-bridge knot $K(r)$, where $r = [2a_1, 2a_2, ...2a_m]$.

4. Theorem 1: Lower and upper bounds on the real part of zeros for two-bridge knots

In this section we prove the following theorem:

Theorem 1. If α is a zero of the Alexander polynomial of a two-bridge knot, then

$$-3 < \mathrm{Re}(\alpha) < 6.$$

Proof. a) To show that $\mathrm{Re}(\alpha) > -k$ we prove that $-(kE + A)$ is stable. Taking $V = E$, it is enough to show that $A_0 = (kE + A) + (kE + A^T) = 2kE + A + A^T$ is positive definite. Now, A_0 is of the form

$$A_0 = \begin{bmatrix} 2k{+}2 & b_1 & & & & & & \\ b_1 & c_1 & b_2 & d_1 & & & & \\ & b_2 & 2k{+}2 & b_3 & & & & \\ & d_1 & b_3 & c_2 & b_4 & d_2 & & \\ & & & b_4 & 2k{+}2 & b_5 & & \\ & & & d_2 & b_5 & c_3 & b_6 & d_3 \\ & & & & & b_6 & 2k{+}2 & b_7 \\ & & & & & & & \ddots \end{bmatrix}$$

where $l = [\frac{m}{2}]$,
$b_j = -\dfrac{1}{a_j} + \dfrac{1}{a_{j+1}}, \ \ j = 1, \ldots, m-1,$
$c_j = (2k+2) - \dfrac{2}{a_{2j-1}a_{2j}} - \dfrac{2}{a_{2j}a_{2j+1}}, \ \ j = 1, \ldots, l-1,$
$c_l = (2k+2) - \dfrac{2}{a_{2l-1}a_{2l}}$ for m even,
$c_l = (2k+2) - \dfrac{2}{a_{2l-1}a_{2l}} - \dfrac{2}{a_{2l}a_{2l+1}}$ for m odd.
$d_j = \dfrac{1}{a_{2j}a_{2j+1}} + \dfrac{1}{a_{2j+1}a_{2j+2}}, \ \ j = 1, \ldots, l-1.$

$$\text{Let } P = \begin{bmatrix} 1 & & & & & \\ -\dfrac{b_1}{2k+2} & 1 & -\dfrac{b_2}{2k+2} & & & \\ & & 1 & & & \\ & & -\dfrac{b_3}{2k+2} & 1 & -\dfrac{b_4}{2k+2} & \\ & & & & 1 & \\ & & & & & \ddots \end{bmatrix}$$

Then

$$PA_0P^T = \begin{bmatrix} 2k+2 & & & & & \\ & \alpha_1 & 0 & \beta_1 & & \\ & 0 & 2k+2 & 0 & & \\ & \beta_1 & 0 & \alpha_2 & 0 & \beta_2 \\ & & & 0 & 2k+2 & 0 \\ & & & \beta_2 & 0 & \ddots \\ & & & \cdots & & \end{bmatrix}$$

$$\approx \begin{bmatrix} 2k+2 & & & 0 \\ & 2k+2 & & \\ & & \ddots & \\ 0 & & & 2k+2 \end{bmatrix} \oplus \begin{bmatrix} \alpha_1 & \beta_1 & & & 0 \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \beta_3 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & & \end{bmatrix}$$

(denote the second matrix by A_{00}),

where

$$\alpha_j = -\frac{b_{2j-1}^2}{2k+2} + c_j - \frac{b_{2j}^2}{2k+2}, \quad j = 1, \dots l-1,$$

$$\alpha_l = \begin{cases} -\frac{b_{2l-1}^2}{2k+2} + c_l & m \text{ is even} \\ -\frac{b_{2l-1}^2}{2k+2} + c_l - \frac{b_{2l}^2}{2k+2} & m \text{ is odd} . \end{cases}$$

$$\beta_j = d_j - \frac{b_{2j} b_{2j+1}}{2k+2}, \quad j = 1, \dots, l-1,$$

We show (i) $\alpha_j > 0$, (ii) $\alpha_j \geq |\beta_{j-1}| + |\beta_j|$, (iii) there exists i such that $\alpha_i > |\beta_{i-1}| + |\beta_i|$. Then A_{00} is positive definite.

Let $k = 3$. Then

$$\alpha_j = 8 - \frac{2}{a_{2j-1}a_{2j}} - \frac{2}{a_{2j}a_{2j+1}} - \frac{1}{8}\left(\frac{-1}{a_{2j-1}} + \frac{1}{a_{2j}}\right)^2 - \frac{1}{8}\left(\frac{-1}{a_{2j}} + \frac{1}{a_{2j+1}}\right)^2$$

$$= 8 - \frac{12}{8}\left(\frac{1}{a_{2j-1}a_{2j}} + \frac{1}{a_{2j}a_{2j+1}}\right) - \frac{1}{8}\left(\frac{1}{a_{2j-1}} + \frac{1}{a_{2j}}\right)^2 - \frac{1}{8}\left(\frac{1}{a_{2j}} + \frac{1}{a_{2j+1}}\right)^2.$$

Since $|a_j| \geq 1$ for all j, $\left|\frac{1}{a_j} + \frac{1}{a_{j+1}}\right| \leq 2$ and $\left|\frac{1}{a_{2j-1}a_{2j}} + \frac{1}{a_{2j}a_{2j+1}}\right| \leq 2$

and hence $\alpha_j \geq 8 - \frac{3}{2} \cdot 2 - \frac{1}{8} \cdot 4 - \frac{1}{8} \cdot 4 = 4.$

On the other hand, $\beta_{j-1} = d_{j-1} - \dfrac{b_{2j-2}b_{2j-1}}{8}$

$$= \frac{1}{a_{2j-2}a_{2j-1}} + \frac{1}{a_{2j-1}a_{2j}} - \frac{1}{8}\left(-\frac{1}{a_{2j-2}} + \frac{1}{a_{2j-1}}\right)\left(-\frac{1}{a_{2j-1}} + \frac{1}{a_{2j}}\right)$$

$$= \frac{7}{8}\left(\frac{1}{a_{2j-2}a_{2j-1}} + \frac{1}{a_{2j-1}a_{2j}}\right) + \frac{1}{8}\left(\frac{1}{a_{2j-2}a_{2j}} + \frac{1}{a_{2j-1}^2}\right).$$

Since $|a_j| \geq 1$, $|\beta_{j-1}| \leq \frac{7}{8} \cdot 2 + \frac{1}{8} \cdot 2 = 2.$

Similarly $|\beta_j| \leq 2$. Thus $\alpha_j \geq |\beta_{j-1}| + |\beta_j|$ and $\alpha_1 > |\beta_1|$. If $\beta_j = 0$ then $\alpha_{j+1} > |\beta_{j+1}|$. This proves the left inequality.

b) To prove that $\mathrm{Re}(\alpha) < q$ it is enough to show that $B_0 = (qE - A) + (qE - A^T) = 2qE - (A + A^T)$ is positive definite. B_0 is of the form

$$B_0 = \begin{bmatrix} 2q-2 & -b_1 & & & & & & & \\ -b_1 & e_1 & -b_2 & -d_1 & & & & & \\ & -b_2 & 2q-2 & -b_3 & & & & & \\ & -d_1 & -b_3 & e_2 & -b_4 & -d_2 & & & \\ & & & -b_4 & 2q-2 & -b_5 & & & \\ & & & -d_2 & -b_5 & e_3 & -b_6 & -d_3 & \\ & & & & & & & \ddots & \end{bmatrix}$$

where $e_j = 2q - 2 + \dfrac{2}{a_{2j-1}a_{2j}} + \dfrac{2}{a_{2j}a_{2j+1}}$, $j = 1, \dots l-1$,

$e_l = 2q - 2 + \dfrac{2}{a_{2l-1}a_{2l}} + \dfrac{2}{a_{2l}a_{2l+1}}$, if m is odd,

$e_l = 2q - 2 + \dfrac{2}{a_{2l-1}a_{2l}}$ if m is even,

($l = [\frac{m}{2}]$, as before.)

Using

$$Q = \begin{bmatrix} 1 & & & & \\ \frac{b_1}{2q-2} & 1 & \frac{b_2}{2q-2} & & \\ & & 1 & & \\ & & \frac{b_3}{2q-2} & 1 & \frac{b_4}{2q-2} \\ & & & & 1 & \ddots \end{bmatrix}$$

we obtain

$$QB_0Q^T = \begin{bmatrix} 2q-2 & & 0 \\ & \ddots & \\ 0 & & 2q-2 \end{bmatrix} \oplus \begin{bmatrix} \gamma_1 & \delta_1 & & \\ \delta_1 & \gamma_2 & \delta_2 & \\ & \delta_2 & \gamma_3 & \delta_3 \\ & & & \ddots \end{bmatrix},$$

where $\gamma_j = e_j - \frac{b_{2j-1}^2}{2q-2} - \frac{b_{2j}^2}{2q-2}, \ j = 1, \dots l$ for m odd,

and γ_l is replaced by $\gamma_l = e_l - \frac{b_{2l-1}^2}{2q-2}$ for m even,

$\delta_j = -d_j - \frac{b_{2j}b_{2j+1}}{2q-2}, \ j = 1, \dots l-1.$

Let $q = 6$. Then

$\gamma_j = 10 + \frac{2}{a_{2j-1}a_{2j}} + \frac{2}{a_{2j}a_{2j+1}} - \frac{1}{10}\left(-\frac{1}{a_{2j-1}} + \frac{1}{a_{2j}}\right)^2 - \frac{1}{10}\left(-\frac{1}{a_{2j}} + \frac{1}{a_{2j+1}}\right)^2$ $\geq 10 - 2 - 2 - \frac{1}{10}\cdot 4 - \frac{1}{10}\cdot 4 = 5.2$, since $\left|-\frac{1}{a_k} + \frac{1}{a_{k+1}}\right| \leq 2$. While $\delta_{j-1} = -\frac{1}{a_{2j-2}a_{2j-1}} - \frac{1}{a_{2j-1}a_{2j}} - \frac{1}{10}\left(-\frac{1}{a_{2j-2}} + \frac{1}{a_{2j-1}}\right)\cdot\left(-\frac{1}{a_{2j-1}} + \frac{1}{a_{2j}}\right)$ and hence $|\delta_{j-1}| \leq \left|\frac{1}{a_{2j-2}a_{2j-1}}\right| + \left|\frac{1}{a_{2j-1}a_{2j}}\right| + \frac{1}{10}\left|\frac{-1}{a_{2j-2}} + \frac{1}{a_{2j-1}}\right|\cdot\left|\frac{-1}{a_{2j-1}} + \frac{1}{a_{2j}}\right| \leq 1 + 1 + \frac{1}{10}\cdot 2\cdot 2 = 2.4$. Also $|\delta_j| \leq 2.4$ and thus $\gamma_j > |\delta_{j-1}| + |\delta_j|$. This proves the right inequality. □

Remark 1. 6 is the best integer upper bound. For the proof see Remark 2 in section 8.

5. Theorem 2: The case of real zeros

Theorem 2. If $a_ja_{j+1} < 0$, then all zeros of $\Delta_K(t)$ are real and positive.

Proof. We show that $\Delta_K(t)$ has a symmetric companion matrix.
Let $r = [2a_1, -2a_2, 2a_3, \dots, (-1)^{m-1}2a_m]$, where $a_j > 0$. Then the Seifert matrix U is of the form

$$U = \begin{bmatrix} a_1 & 0 & & & \\ -1 & -a_2 & 1 & & \\ & 0 & a_3 & 0 & \\ & & -1 & -a_4 & 1 \\ & & & & a_5 \\ & & & & \ddots \end{bmatrix}$$

Now

$$Ut - U^T = \begin{bmatrix} a_1(t-1) & 1 & & & \\ -t & -a_2(t-1) & t & & \\ & -1 & a_3(t-1) & 1 & \\ & & & \ddots & \end{bmatrix}$$

We apply a series of transformations that don't change the zeros of the determinant of the matrix. First, multiply -1 on all even rows to get

$$\begin{bmatrix} a_1(t-1) & 1 & & & \\ t & a_2(t-1) & -t & & \\ & -1 & a_3(t-1) & 1 & \\ & & & \ddots & \end{bmatrix}$$

Then multiply $\dfrac{1}{\sqrt{a_1}}$ on the 1-st row and column, $\dfrac{1}{\sqrt{a_2}}$ on the 2-nd row and column, and so on, to get

$$M = \begin{bmatrix} t-1 & \frac{1}{\sqrt{a_1a_2}} & & & \\ \frac{t}{\sqrt{a_1a_2}} & t-1 & \frac{-t}{\sqrt{a_2a_3}} & & \\ & \frac{-1}{\sqrt{a_2a_3}} & t-1 & \frac{1}{\sqrt{a_3a_4}} & \\ & & & \ddots & \end{bmatrix}$$

with $\det(M) = \dfrac{1}{a_1a_2\cdots a_m}\det(Ut-U^t)$. Now eliminate t from the off-diagonal line as follows: multiply $-\dfrac{1}{\sqrt{a_1a_2}}$ on the 1-st row and add it to the 2-nd row, multiply $\dfrac{1}{\sqrt{a_2a_3}}$ on the 3-rd row and add it to the second row, multiply $-\dfrac{1}{\sqrt{a_3a_4}}$ on the 3-rd row and add it to the 5-th row, etc, i.e. multiply M by matrix P from the left:

$$P = \begin{bmatrix} 1 & 0 & & & & \\ -\frac{1}{\sqrt{a_1a_2}} & 1 & \frac{1}{\sqrt{a_2a_3}} & & & \\ & 0 & 1 & 0 & & \\ & & -\frac{1}{\sqrt{a_3a_4}} & 1 & \frac{1}{\sqrt{a_4a_5}} & \\ & & & 0 & 1 & \\ & & & & & \ddots \end{bmatrix}$$

Then $PM =$

$$\begin{bmatrix} t-1 & \frac{1}{\sqrt{a_1a_2}} & 0 & 0 & & & \\ \frac{1}{\sqrt{a_1a_2}} & (t-1-\frac{1}{a_1a_2}-\frac{1}{a_2a_3}) & \frac{-1}{\sqrt{a_2a_3}} & \frac{1}{\sqrt{a_2a_3}\sqrt{a_3a_4}} & 0 & & \\ 0 & \frac{-1}{\sqrt{a_2a_3}} & t-1 & \frac{1}{\sqrt{a_3a_4}} & 0 & 0 & \\ 0 & \frac{1}{\sqrt{a_2a_3}\sqrt{a_3a_4}} & \frac{1}{\sqrt{a_3a_4}} & (t-1-\frac{1}{a_3a_4}-\frac{1}{a_4a_5}) & \frac{-1}{\sqrt{a_4a_5}} & \frac{1}{\sqrt{a_4a_5}\sqrt{a_5a_6}} & \\ & 0 & 0 & \frac{-1}{\sqrt{a_4a_5}} & t-1 & \frac{1}{\sqrt{a_5a_6}} & \\ & & & & & & \ddots \end{bmatrix}$$

Since $PM = tE - A$, A is a companion matrix of $\Delta_K(t)$ and it is symmetric. So all its eigenvalues are real, and hence positive. $\square$

6. Theorem 3: The case $a_ia_{i+1} \neq 1$

Theorem 3. Let $r = [2\varepsilon_1a_1, 2\varepsilon_2a_2, \dots, 2\varepsilon_ma_m]$, where $a_i > 0$, $\varepsilon_i = \pm 1$. If we don't have $a_i = a_{i+1} = 1$ and $\varepsilon_i = \varepsilon_{i+1}$, then the zeros of $\Delta_{K(r)}$ satisfy inequality:

$$-1 < \mathrm{Re}(\alpha).$$

If, moreover, $a_j > 1$ for all j, then $\mathrm{Re}(\alpha) < 3$.

Proof. We find a positive definite (symmetric) matrix V such that $V(E+A) + (E+A^T)V = W$ is positive definite. Let V be a diagonal matrix with elements $a_1, a_2, \dots, a_m$. Then multiplying $(E+A)$ by V from the left is multiplying the $i-$th row of $(E+A)$ by a_i, $i = 1, \dots, m$. Let $l = [\frac{m}{2}]$. Define $\varepsilon_{ij} = \varepsilon_i\varepsilon_j$.

By (3.3) we have $E + A =$

$$= \begin{bmatrix} 2 & -\frac{\varepsilon_1}{a_1} & 0 & & \dots & & & \\ \frac{\varepsilon_2}{a_2} & 2-\frac{\varepsilon_{12}}{a_1a_2}-\frac{\varepsilon_{23}}{a_2a_3} & -\frac{\varepsilon_2}{a_2} & \frac{\varepsilon_{23}}{a_2a_3} & 0 & \dots & & \\ 0 & \frac{\varepsilon_3}{a_3} & 2 & -\frac{\varepsilon_3}{a_3} & 0 & 0 & 0 & \dots \\ 0 & \frac{\varepsilon_{34}}{a_3a_4} & \frac{\varepsilon_4}{a_4} & 2-\frac{\varepsilon_{34}}{a_3a_4}-\frac{\varepsilon_{45}}{a_4a_5} & -\frac{\varepsilon_4}{a_4} & \frac{\varepsilon_{45}}{a_4a_5} & 0 & \dots \\ 0 & 0 & 0 & \frac{\varepsilon_5}{a_5} & 2 & -\frac{\varepsilon_5}{a_5} & 0 & \dots \\ & & & \dots & & & & \end{bmatrix}$$

where the last row is $(0, \dots, 0, \frac{\varepsilon_m}{a_m}, 2)$ if m is odd,
and $(0, \dots, 0, \frac{\varepsilon_{m-1,m}}{a_{m-1}a_m}, \frac{\varepsilon_m}{a_m}, 2-\frac{\varepsilon_{m-1,m}}{a_{m-1}a_m},)$ if m is even.

Therefore $V(E+A) =$

$$
= \begin{bmatrix}
2a_1 & -\varepsilon_1 & 0 & & \dots & & & \\
\varepsilon_2 & 2a_2 - \frac{\varepsilon_{12}}{a_1} - \frac{\varepsilon_{23}}{a_3} & -\varepsilon_2 & \frac{\varepsilon_{23}}{a_3} & 0 & \dots & & \\
0 & \varepsilon_3 & 2a_3 & -\varepsilon_3 & 0 & 0 & 0 & \dots \\
0 & \frac{\varepsilon_{34}}{a_3} & \varepsilon_4 & 2a_4 - \frac{\varepsilon_{34}}{a_3} - \frac{\varepsilon_{45}}{a_5} & -\varepsilon_4 & \frac{\varepsilon_{45}}{a_5} & 0 & \dots \\
0 & 0 & 0 & \varepsilon_5 & 2a_5 & -\varepsilon_5 & 0 & \dots \\
 & & & \dots & & & & \\
 & & & & & & &
\end{bmatrix}
$$

where the last row is $(0, \dots, 0, \varepsilon_m, 2a_m)$ if m is odd, and $(0, \dots, 0, \varepsilon_m, 2a_m - \frac{\varepsilon_{m-1,m}}{a_{m-1}})$ if m is even. Further $W = V(E+A) + (E+A^T)V =$

$$
\begin{bmatrix}
4a_1 & -\varepsilon_1+\varepsilon_2 & 0 & & \dots & & \\
-\varepsilon_1+\varepsilon_2 & 4a_2 - \frac{2\varepsilon_{12}}{a_1} - \frac{2\varepsilon_{23}}{a_3} & -\varepsilon_2+\varepsilon_3 & \frac{\varepsilon_{23}+\varepsilon_{34}}{a_3} & \dots & & \\
0 & -\varepsilon_2+\varepsilon_3 & 4a_3 & -\varepsilon_3+\varepsilon_4 & 0 & 0 & \dots \\
0 & \frac{\varepsilon_{23}+\varepsilon_{34}}{a_3} & -\varepsilon_3+\varepsilon_4 & 4a_4 - \frac{2\varepsilon_{34}}{a_3} - \frac{2\varepsilon_{45}}{a_5} & -\varepsilon_4+\varepsilon_5 & \frac{\varepsilon_{45}+\varepsilon_{56}}{a_5} & 0 \\
0 & 0 & 0 & -\varepsilon_4+\varepsilon_5 & 4a_5 & -\varepsilon_5+\varepsilon_6 & 0 \\
0 & 0 & 0 & \frac{\varepsilon_{45}+\varepsilon_{56}}{a_5} & -\varepsilon_5+\varepsilon_6 & 4a_6 - \frac{2\varepsilon_{56}}{a_5} - \frac{2\varepsilon_{67}}{a_7} & \dots \\
 & & & \dots & & & \ddots
\end{bmatrix}
$$

The last row of W is $(\ 0, \dots, 0, -\varepsilon_{m-1}+\varepsilon_m,\ 4a_m\ )$ if m is odd, and $(\ 0, \dots, 0, \dfrac{\varepsilon_{m-2,m-1}+\varepsilon_{m-1,m}}{a_{m-1}},\ -\varepsilon_{m-1}+\varepsilon_m,\ 4a_m - \dfrac{2\varepsilon_{m-1,m}}{a_{m-1}}\ )$ if m is even.

We eliminate the elements $-\varepsilon_i + \varepsilon_{i+1}$: if i is odd, multiply the i-th row by $(\varepsilon_i - \varepsilon_{i+1})/4a_i$ and add to the $(i+1)$-th row. If i is even, multiply the $(i+1)$-th row by $(\varepsilon_i - \varepsilon_{i+1})/4a_{i+1}$ and add to the i-th row. Similarly for columns. In other words we consider the matrix PWP^T, where

$$
P = \begin{bmatrix}
1 & & & & & \\
\frac{\varepsilon_1-\varepsilon_2}{4a_1} & 1 & \frac{\varepsilon_2-\varepsilon_3}{4a_3} & & & \\
 & & 1 & & & \\
 & & \frac{\varepsilon_3-\varepsilon_4}{4a_3} & 1 & \frac{\varepsilon_4-\varepsilon_5}{4a_5} & \\
 & & & & 1 & \\
 & & & & & \ddots
\end{bmatrix}
$$

We have

$$
PWP^T = \begin{bmatrix}
4a_1 & 0 & 0 & & & & \\
0 & \alpha_2 & 0 & \beta_2 & 0 & 0 & \\
0 & 0 & 4a_3 & 0 & 0 & & \\
0 & \beta_2 & 0 & \alpha_4 & 0 & \beta_4 & \dots \\
 & & & 0 & 4a_5 & 0 & \dots \\
 & & & \beta_4 & 0 & \alpha_6 & \dots \\
 & & & & & & \dots
\end{bmatrix} \sim
$$

$$\sim \begin{bmatrix} 4a_1 & & & \\ & 4a_3 & & \\ & & \ddots & \\ & & & 4a_{2l\pm1} \end{bmatrix} \oplus \begin{bmatrix} \alpha_2 & \beta_2 & & & \\ \beta_2 & \alpha_4 & \beta_4 & 0 & \\ & \beta_4 & \alpha_6 & \beta_6 & \dots \\ & & & \ddots & \\ & & & \beta_{2l-2} & \alpha_{2l} \end{bmatrix} \tag{6.1}$$

Here $l = [\frac{m}{2}]$, and for $i = 1, \dots, l-1$

$$\begin{aligned} \alpha_{2i} &= 4a_{2i} - \frac{2\varepsilon_{2i-1,2i}}{a_{2i-1}} - \frac{2\varepsilon_{2i,2i+1}}{a_{2i+1}} - \frac{(\varepsilon_{2i-1}-\varepsilon_{2i})^2}{4a_{2i-1}} - \frac{(\varepsilon_{2i}-\varepsilon_{2i+1})^2}{4a_{2i+1}} \\ &= 4a_{2i} - \frac{3}{2}\frac{\varepsilon_{2i-1,2i}}{a_{2i-1}} - \frac{3}{2}\frac{\varepsilon_{2i,2i+1}}{a_{2i+1}} - \frac{1}{2a_{2i-1}} - \frac{1}{2a_{2i+1}}, \end{aligned} \tag{6.2}$$

$$\begin{aligned} \beta_{2i} &= \frac{\varepsilon_{2i,2i+1}+\varepsilon_{2i+1,2i+2}}{a_{2i+1}} - \frac{(\varepsilon_{2i}-\varepsilon_{2i+1})(\varepsilon_{2i+1}-\varepsilon_{2i+2})}{4a_{2i+1}} \\ &= \frac{3}{4}\frac{(\varepsilon_{2i,2i+1}+\varepsilon_{2i+1,2i+2})}{a_{2i+1}} + \frac{\varepsilon_{2i,2i+2}+1}{4a_{2i+1}}, \end{aligned}$$

$$\alpha_{2l} = 4a_{2l} - \frac{3}{2}\frac{\varepsilon_{2l-1,2l}}{a_{2l-1}} - \frac{3}{2}\frac{\varepsilon_{2l,2l+1}}{a_{2l+1}} - \frac{1}{2a_{2l-1}} - \frac{1}{2a_{2l+1}}, \quad \text{if } m \text{ is odd},$$

$$\alpha_{2l} = 4a_{2l} - \frac{3}{2}\frac{\varepsilon_{2l-1,2l}}{a_{2l-1}} - -\frac{1}{2a_{2l-1}}, \quad \text{if } m \text{ is even}.$$

Since all $a_j \geq 1$, it is not difficult to check that if among $\frac{\varepsilon_{2i-1}}{a_{2i-1}}, \frac{\varepsilon_{2i}}{a_{2i}}, \frac{\varepsilon_{2i+1}}{a_{2i+1}}$, there are no two consecutive 1 or -1, then the conditions of Positivity Lemma are satisfied: (i) $\alpha_{2i} > 0$, (ii) $\alpha_{2i} \geq |\beta_{2i-2}|+|\beta_{2i}|$, $i = 2, \dots, l-1$, and (iii) $\alpha_2 > |\beta_2|$. If $\beta_{2j} = 0$ then $\alpha_{2j+2} > |\beta_{2j+2}|$. So the second matrix in (6.1) is positive definite and so is W.

The proof of inequality $\mathrm{Re}(\alpha) < 3$ in the case $a_j > 1$ for $j = 1, 2, \dots, m$, is similar to the proof of Theorem 1 for $q = 3$. □

7. Theorem 4: The case of fibered knots

Consider a fibered two-bridge knot $K(r)$ with $r = [2a_1, 2a_2, \dots, 2a_m]$

$$= [\ \underbrace{2,\dots,2,}_{k_1}\ \underbrace{-2,\dots,-2,}_{k_2}\ \dots,\ \underbrace{(-1)^{m-1}2,\dots,(-1)^{m-1}2}_{k_m}\ ].$$

The following theorem is a corollary of Theorem 3:

Theorem 4. If $k_j = 1$ or 2, $j = 1, \dots m$, then $-1 < \mathrm{Re}(\alpha)$.

Proof. At least one of $\varepsilon_{2i-1,2i}$, $\varepsilon_{2i,2i+1}$ in (6.2) is negative. So by (6.2)

$$\alpha_{2i} = \begin{cases} 3 & \text{if } \varepsilon_{2i-1} \neq \varepsilon_{2i+1} \\ 6 & \text{if } \varepsilon_{2i-1} = \varepsilon_{2i+1} \neq \varepsilon_{2i} \end{cases}$$

While

$$\beta_{2i} = \begin{cases} 0 & \text{if } \varepsilon_{2i} \neq \varepsilon_{2i+2} \\ -1 & \text{if } \varepsilon_{2i} = \varepsilon_{2i+2} \neq \varepsilon_{2i+1} \end{cases}$$

and similarly $\beta_{2i-2} = 0$ or -1. So the conditions of Positivity Lemma are satisfied, which proves the inequality. □

8. Theorem 5: The case $a_i = \pm c$

Theorem 5. Let $r_m = [2c, -2c, \ldots, (-1)^{m-1}2c]$, $c > 0$, $m \geq 1$. Then all zeros of $\Delta_{K(r_m)}$ satisfy inequality:

$$\left(\frac{\sqrt{1+c^2}-1}{c}\right)^2 < \alpha < \left(\frac{\sqrt{1+c^2}+1}{c}\right)^2.$$

Proof. By (3.1) a Seifert matrix for $K(r_m)$ is

$$U = \begin{bmatrix} c & 0 & & & & \\ -1 & -c & 1 & & & \\ & 0 & c & 0 & & \\ & & -1 & -c & 1 & \\ & & & & & \ddots \end{bmatrix}$$

Let $P_0(t) = 1$, $P_1(t) = c(t-1)$, $P_m(t) = (-1)^{[\frac{m}{2}]}\det(tU - U^T) =$

$$= (-1)^{[\frac{m}{2}]}\det \begin{bmatrix} c(t-1) & 1 & & & & \\ -t & c(-t+1) & t & & & \\ & -1 & c(t-1) & 1 & & \\ & & -t & c(-t+1) & t & \\ & & & & & \ddots \end{bmatrix} =$$

$$= \det \begin{bmatrix} c(t-1) & 1 & & & \\ t & c(t-1) & -t & & \\ & -1 & c(t-1) & 1 & \\ & & t & c(t-1) & \\ & & & & \ddots \end{bmatrix}$$

Then $P_m(t) = \pm\Delta_{K(r_m)}(t)$, and $P_m(t)$ satisfy a recurrence equation:

$$P_m(t) = c(t-1)P_{m-1}(t) - tP_{m-2}(t),\ m \geq 2. \tag{8.1}$$

Since $K(r_{2m+1})$ is a 2-component link, we can write $P_{2m+1}(t) = (t-1)Q_{2m}(t)$. Note $Q_0(t) = c$. Then from (8.1) we have

$$P_{2m}(t) = c(t-1)^2 Q_{2m-2}(t) - tP_{2m-2}(t). \tag{8.2}$$

Also,

$$P_{2m+1}(t) = c(t-1)P_{2m}(t) - tP_{2m-1}(t) \Longrightarrow$$
$$(t-1)Q_{2m}(t) = c(t-1)P_{2m}(t) - t(t-1)Q_{2m-2}(t) \Longrightarrow$$

$$Q_{2m}(t) = cP_{2m}(t) - tQ_{2m-2}(t). \tag{8.3}$$

Then (8.2) and (8.3) imply

$$t^{-m}P_{2m}(t) = t^{-m}c(t-1)^2 Q_{2m-2}(t) - t^{-(m-1)}P_{2m-2}(t)$$

and

$$t^{-m}Q_{2m}(t) = ct^{-m}P_{2m}(t) - t^{-(m-1)}Q_{2m-2}(t).$$

Let $x = t + \dfrac{1}{t}$, and write $\phi_m(x) = t^{-m}P_{2m}(t)$, $\psi_m(x) = t^{-m}Q_{2m}(t)$. Then

$$\phi_m(x) = c(x-2)\psi_{m-1}(x) - \phi_{m-1}(x), \tag{8.4}$$

$$\psi_m(x) = c\phi_m(x) - \psi_{m-1}(x). \tag{8.5}$$

Note $\phi_0(x) = 1,\ \psi_0(x) = c$. Since (8.4) $\Longrightarrow c(x-2)\psi_{m-1}(x) = \phi_m(x) + \phi_{m-1}(x)$, from (8.5) we see:

$$c(x-2)\psi_m(x) = c^2(x-2)\phi_m(x) - c(x-2)\psi_{m-1}(x) \Longrightarrow$$

$$\phi_{m+1}(x) + \phi_m(x) = c^2(x-2)\phi_m(x) - (\phi_m(x) + \phi_{m-1}(x)) \Longrightarrow$$

$$\phi_{m+1}(x) = (c^2x - (2c^2+2))\phi_m(x) - \phi_{m-1}(x) \tag{8.6}$$

Similarly, using (8.4) and (8.5), we have

$$\psi_m(x) = (c^2x - (2c^2+2))\psi_{m-1}(x) - \psi_{m-2}(x) \tag{8.7}$$

Let $y = c^2x - (2c^2+2)$. Write $\phi_m(x) = \lambda_m(y)$ and $\psi_m(x) = \mu_m(y)$. Then from (8.6) and (8.7) we have, for $m \geq 2$,

$$\lambda_m(y) = y\lambda_{m-1}(y) - \lambda_{m-2}(y)$$

$$\mu_m(y) = y\mu_{m-1}(y) - \mu_{m-2}(y),$$

where $\lambda_0 = 1,\ \lambda_1 = y+1,\ \lambda_2 = y^2+y-1,\ \mu_0 = c,\ \mu_1 = cy,\ \mu_2 = c(y^2-1)$. It is easy to see that for $m \geq 1$, $\lambda_m = \dfrac{1}{c}(\mu_m + \mu_{m-1})$. Now let $f_m(y)$ be a Fibonacci polynomial defined in [K]: $f_1(y) = 1,\ f_2(y) = y$ and for $m \geq 3$,

$$f_m(y) = yf_{m-1}(y) + f_{m-2}(y).$$

Then we can show by induction that for $m \geq 0$,

$$i^{-m} f_{m+1}(iy) = \frac{1}{c}\mu_m(y).$$

It is known (see [K], p.477) that the zeros of $f_{m+1}(y)$ are $y_k = 2i\cos\frac{k\pi}{m+1}$, $k = 1, 2, \ldots, m$. Therefore, the zeros of $\mu_m(y)$ are

$$y_k^{(m)} = 2\cos\frac{k\pi}{m+1},\quad k = 1, 2, \ldots, m.$$

Next we look at the zeros of $\lambda_m(y)$. Since $y_k^{(m-1)} = 2\cos\frac{k\pi}{m},\quad k = 1, 2, \ldots, m-1$, are all the zeros of $\mu_{m-1}(y)$, and for any k

$$y_{k+1}^{(m-1)} < y_{k+1}^{(m)} < y_k^{(m-1)} < y_k^{(m)}, \tag{8.8}$$

there exists exactly one zero of $\mu_m(y)$ between neighboring two zeros of $\mu_{m-1}(y)$, and also there exists exactly one zero of $\mu_{m-1}(y)$ between neighboring two zeros of $\mu_m(y)$ (see Fig.3). By induction we check that

$$\mu_{2m}(-2) = (2m+1)c \quad \text{and} \quad \mu_{2m+1}(-2) = -(2m+2)c. \tag{8.9}$$

Now, the zeros of $\lambda_m(y)$ occur at the intersections of two curves $c_1:\ z = (-1)^m\mu_m(y)$ and $c_2:\ z = (-1)^{m-1}\mu_{m-1}(y)$. By (8.8) there are $m-1$ zeros in $(y_{m-1}^{(m)}, 2)$, and by (8.9) two curves intersect in $(-2, y_{m-1}^m)$. Therefore there are exactly m real zeros in $(-2, 2)$. Since $y = c^2x - (2c^2+2)$, $x = \dfrac{y + (2c^2+2)}{c^2}$ and the zeros of $\phi_m(x)$ and $\psi_m(x)$ are in the interval $(2,\ 2 + \dfrac{4}{c^2})$, and hence all zeros of $P_{2m}(t)$ and $Q_{2m}(t)$ satisfy inequality:

$$\frac{1}{q} = (\frac{\sqrt{1+c^2}-1}{c})^2 < \alpha < q = (\frac{\sqrt{1+c^2}+1}{c})^2. \qquad \square$$

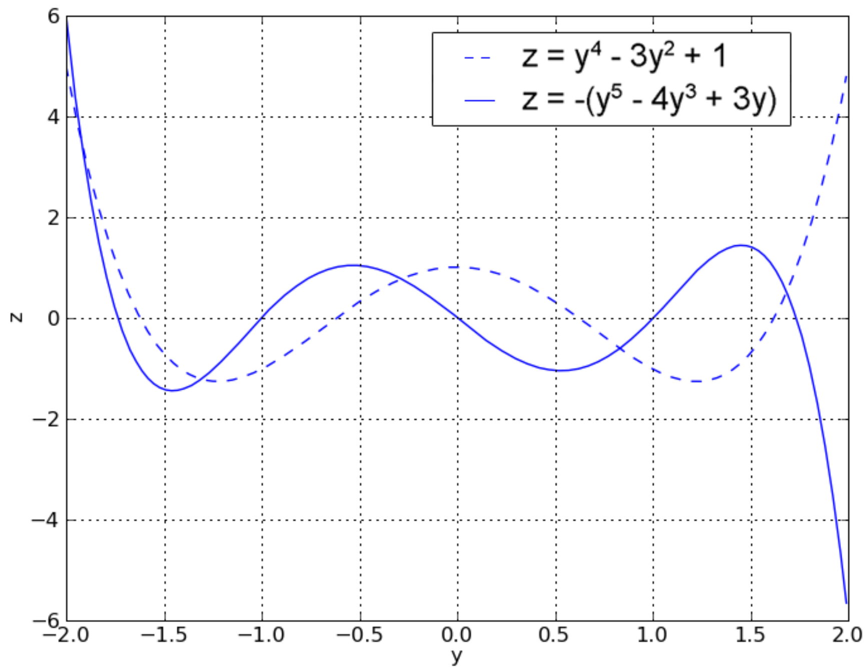

Figure 3.

Corollary. *If* $c \to \infty$, *then the zeros of* $P_{2m}(t)$, $Q_{2m}(t)$, *which are the zeros of Alexander polynomials, tend to* 1.

Remark 2. For $c = 1$ and large enough m we can find a zero α of $P_{2m}(t)$ arbitrarily close to $q = 3 + \sqrt{8}$. It is quite likely that $3 + \sqrt{8}$ is the upper bound of the real part of the zeros.

Proof. Since the zeros of $(-1)^{m-1}\mu_{m-1}(y)$ and $(-1)^m\mu_m(y)$ satisfy inequality $y_2^{(m)} < y_1^{(m-1)} < y_1^{(m)}$, there is a zero of λ_m greater than $y_2^{(m)}$, where $y_2^{(m)} = 2\cos\dfrac{2\pi}{m+1}$. So there is a zero of $\phi_m(x)$ arbitrarily close to 6, hence a zero of $P_{2m}(t)$ arbitrarily close to $3 + \sqrt{8}$. $\square$

9. Open questions

Let us finish with several open questions:

1) Is there an upper bound of the real part of zeros of the Alexander polynomials of general alternating knots ? Recently Hirasawa observed(2010) that each of the following alternating 12 crossing knots $12a_{0125}$ and $12a_{1124}$ has a real zero, 6.90407... and 7.69853... respectively. Therefore an upper bound, if exists, is larger than 7.

2) Given m, does there exist an upper bound q(m) of the real part of zeros of the Alexander polynomials of degree m of alternating knots ?

3) Is there a version of Conjecture 1 for non-alternating knots ?

Notice that Conjecture 1 does not hold for homogeneous knots (defined in [Cr]). Hirasawa showed (2010) that a non-alternating knot 10_{152} is a closure of a positive 3-braid and hence it is a homogeneous knot, but the Alexander polynomial has a real zero $\alpha = -1.85...$

4) Characterize alternating knots whose zeros of the Alexander polynomial are real. In particular, is the converse of Theorem 2 true for one component two-bridge knots?

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